The ideal of Sierpiński-Zygmund sets on the plane

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Abstract

We say that a set $X \subseteq \mathbb{R}^2$ is *Sierpiński-Zygmund* (shortly *SZ-set*) if it does not contain a partial continuous function of cardinality continuum $\mathfrak{c}$. We observe that the family of all such sets is $\mathrm{cf}(\mathfrak{c})$-additive ideal. Some examples of such sets are given. We also consider *SZ-shiftable sets*, that is, sets $X \subseteq \mathbb{R}^2$ for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f + X$ is an SZ-set. Some results are proved about SZ-shiftable sets. In particular, we show that the union of two SZ-shiftable sets does not have to be SZ-shiftable.

The terminology is standard and follows [2]. The symbol $\mathbb{R}$ stands for the set of all real numbers. The cardinality of a set X we denote by $|X|$. In particular, $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. Given a cardinal κ, we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular provided that $\mathrm{cf}(\kappa) = \kappa$.

A set $M \subseteq \mathbb{R}^n$ is called *Marczewski measurable* if every perfect set P has a perfect subset Q such that $Q \subseteq M$ or $Q \cap M = \emptyset$. If every perfect set P has a perfect subset Q such that $Q \cap M = \emptyset$, then M is called *Marczewski null*.

We consider only real-valued functions unless stated otherwise. No distinction is made between a function and its graph. For any planar set Y, we denote its x-projection by $\mathrm{dom}(Y)$. For any two partial real functions f, g we write $f + g$, $f - g$

$g \quad \mathbb{R}^X$, any family of functions $F \quad \mathbb{R}^X$, and any set $A \quad X \times \mathbb{R}$ we define $g + F = \{g + f : f \quad F\}$ and $g + A = \{ x, g(x) + y : x, y \quad A\}$. The image and preimage of a set B under the function h are denoted by $h[B]$ and $h^{-1}[B]$, respectively.

Let us recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *Sierpiński-Zygmund* ($f \quad$ SZ) if for every set $X \quad \mathbb{R}$ of cardinality continuum $\mathfrak{c}$, $f|X$ is discontinuous. This definition is generalized onto subsets of $\mathbb{R}^2$. (See [8].)

Definition 1 A set $X \quad \mathbb{R}^2$ is called *Sierpiński-Zygmund* set (shortly *SZ-set*), if for every partial real continuous function f we have $|f \cap X| < \mathfrak{c}$.

We denote the family of all SZ-sets by J_{SZ}. Since every Sierpiński-Zygmund function is also an SZ-set we have that the family J_{SZ} is not empty.

The next fact follows directly from the definition.

Fact 2 J_{SZ} is a cf($\mathfrak{c}$)-additive ideal.

Proof. It is obvious that J_{SZ} is closed under the operation of taking subsets. We will show that *szsets* is cf($\mathfrak{c}$)-additive.

Take a $\kappa < \mathrm{cf}(\mathfrak{c})$. Let $\{X_\xi : \xi < \kappa\} \quad J_{SZ}$ and $f \quad \bigcup_{\xi<\kappa} X_\xi$ be a partial continuous function. Since X_ξ is SZ-set, we have that $|f \cap X_\xi| < \mathfrak{c}$ for each $\xi < \kappa$. Consequently, $|f \cap \bigcup_{\xi<\kappa} X_\xi| = |\bigcup_{\xi<\kappa}(f \cap X_\xi)| < \mathfrak{c}$. ∎

The question that one could ask here is how “big” an SZ-set can be. An example of the SZ-set that can be considered “big” in some sense is given in [8].

Lemma 3 [8, Lemma 19] *There exists an SZ-set $X \quad \mathbb{R}^2$ such that $|\mathbb{R} \setminus X_x| < \mathfrak{c}$ for every*

We claim that there exists an $A \quad [\mathbb{R}]^{\omega_1}$ such that $|\bigcup_{y \quad A} X^y| < \mathfrak{c}$. The following two cases are possible.

Case 1. There exists a $\kappa < \mathfrak{c}$ such that $Z_\kappa = \{y\colon |X^y| = \kappa\}$ is uncountable.

Then we choose $A \quad [Z_\kappa]^{\omega_1}$. Obviously, $|\bigcup_{y \quad A} X^y| = \kappa\omega_1 < \mathfrak{c}$.

Case 2. $|Z_\kappa| \leq \omega$ for every cardinal $\kappa < \mathfrak{c}$.

Put $Z = \{|X^y|\colon y \quad \mathbb{R}\}$ and observe that $\mathbb{R} = \bigcup_{\kappa \quad Z} Z_\kappa$. It follows from ( ) that if $\kappa \quad Z$ then $\kappa < \mathfrak{c}$. Consequently, since the union of less than continuum many countable sets has size less than continuum, we conclude that $|Z| = \mathfrak{c}$. Let λ be the ω_1-st element of Z. We define $A = \{y\colon \quad [Z_\kappa$

any more. However, if $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism preserving vertical lines then $h[X]$ is an SZ-set for every $X \quad J_{SZ}$.

Fact 6 Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an homeomorphism such that $h[L]$ is a vertical line for every vertical line L. Then $h\{J_{SZ}\} = \{h[X]\colon X \quad J_{SZ}\} = J_{SZ}$.

Proof. First we show the inclusion $h\{J_{SZ}\} \quad J_{SZ}$. It is easy to see that if $f : A \to \mathbb{R}$ is a partial continuous function then $h^{-1}[f]\colon A \to \mathbb{R}$ is also continuous. This implies that for every $X \quad J_{SZ}$, $h[X]$ is also in J_{SZ}.

Now to show the other inclusion, let us fix a $Y \quad J_{SZ}$. Note that h^{-1} also preserves all vertical lines. Thus, from the first part of the proof, $X = h^{-1}[Y] \quad J_{SZ}$. Hence $Y = h[X] \quad h\{J_{SZ}\}$. ■

As we mentioned at the beginning of this paper, the concept of Sierpiński-Zygmund sets is a generalization of the concept of Sierpiński-Zygmund functions. One of the questions related to the family SZ of Sierpiński-Zygmund functions is for how “big” families $F \quad \mathbb{R}^{\mathbb{R}}$ we can find a function $g \quad \mathbb{R}^{\mathbb{R}}$ such that $g + F \quad$ SZ. (See e.g. [3].) Similar question can be asked in the case of Sierpiński-Zygmund sets. This leads to the following definition.

Definition 7 A set $X \quad \mathbb{R}^2$ is called SZ-*shiftable*, if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f + X$ is SZ-set.

We denote the family of all SZ-shiftable sets by SZ_{shift}. Obviously $J_{SZ} \quad SZ_{shift}$, so SZ_{shift} is not empty.

Lemma 8 *Let* $X \quad \mathbb{R}^2$*. If for all* $x \quad \mathbb{R}$ *and* $A \quad [\mathbb{R}]^{<\mathfrak{c}}$ *there exists an* $a \quad \mathbb{R}$ *such that* $(a + A) \cap X_x = \quad$*, then* A *is SZ-shiftable.*

Proof. Let $x_\alpha : \alpha < \mathfrak{c}$ and $f_\alpha : \alpha < \mathfrak{c}$ be the sequences of all real numbers and all continuous functions defined on a G_δ subset of $\mathbb{R}$, respectively. We will define a function $f : \mathbb{R} \to \mathbb{R}$ which shifts X into J_{SZ}, using transfinite induction. For every $\alpha < \mathfrak{c}$ we choose $f(x_\alpha) \quad \mathbb{R}$ with the property that ep309.963 Tf15.74 0 Td[3Tf .27 0 Td[(an28 0 Td[(w)28(e)-357(c)

It may also be of interest to determine whether SZ_{shift} is closed under the union operation. Fact 2 states, in particular, that the union of two SZ-sets is also an SZ-set. Thus, the natural question that appears here is whether the same is true for SZ-shiftable sets. It turns out not to be the case.

Example 10 *There exist $A_1, A_2 \in SZ_{shift}$ such that $A_1 \cup A_2 = \mathbb{R}^2 \notin SZ_{shift}$.*

PROOF. Put A_1 to be the set X from Lemma 3 and A_2 to be its complement. Based on Lemma 8 A_2 is SZ-shiftable. Next, notice that $A_1 \in J_{SZ} \subset SZ_{shift}$. Finally, $A_1 \cup A_2 = \mathbb{R}^2$ and obviously $\mathbb{R}^2$ is not in SZ_{shift}. ■

Before we finish let us make a comment about [8, Theorem 2 (1)] which says: MA implies that for every finite family F of real functions there exists an almost continuous function g (each open subset of $\mathbb{R}^2$ containing the graph of g contains also the graph of a continuous function) such that $g + f$ is Sierpiński-Zygmund for every $f \in F$. Note that this result can be expressed using the notion of SZ-sets. Under MA the following holds:

> *If, for some fixed $n \in \omega$, every vertical section of the set $X \subset \mathbb{R}^2$ has at most n elements then there exists an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f + X \in J_{SZ}$.*

We generalize the above result.

Theorem 11

Lemma 13 (MA) *Let $F \subseteq F_A$ be a family such that $|F| \leq \mathfrak{c}$. There exists a $g \in$ SZ(A) such that $g + F \subseteq$ SZ(A) and for every blocking set $B \subseteq \mathbb{R}^2$ there is a non-empty open interval $I_B \subseteq$ dom(B) with the property that* dom$(B \cap g)$ *is dense in I_B.*

Lemma 14 [8, Lemma 13] (MA) *Let $\{f_i\}_1^n \subseteq \mathbb{R}^{\mathbb{R}}$, $n = 1, 2, \ldots$. There exists $\{f_i\}_1^n \subseteq F_A$ such that $f_i|A_i \in$ C$_{<\mathfrak{c}}(A_i)$, where $A_i = [f_i = f_i]$.*

Note that Lemmas 13 and 14 imply the following.

(∗) (MA) Assume that $F \subseteq \mathbb{R}^{\mathbb{R}}$ is finite and $A \subseteq \mathbb{R}$ is everywhere of second category. Then there exists a function $g: A \to \mathbb{R}$ such that $g + F \subseteq$ SZ(A) and dom$(g \cap B)$ is dense in some non-empty open interval I_B for every blocking set B.

Proof. Let us consider the partition $\{H_n : n \in \omega\}$ of $\mathbb{R}$, where H_n is defined by $H_n = \{x \in \mathbb{R}: |X_x| = n\}$. Let $G_n \subseteq \mathbb{R}$ be a maximal open set such that H_n is everywhere of second category in G_n. Such a set can be easily constructed. Simply define G_n as the interior of the set $\mathbb{R} \setminus \bigcup_{I \in \mathcal{I}_n} I$, where $\mathcal{I}_n$ is the set of all open intervals in which H_n is meager.

We claim that for every $n < \omega$, there exists a function $g_n: (G_n \cap H_n) \to \mathbb{R}$ such that $g_n + X = \{\langle x, g_n(x) + y\rangle : x \in (G_n \cap H_n), \langle x, y\rangle \in X\} \in J_{SZ}$ and $\bigcup_{n<\omega} g_n$ intersects every blocking set B.

First observe that this claim implies the conclusion of the theorem. Put $g: \mathbb{R} \to \mathbb{R}$ to be an extension of $\bigcup_{n<\omega} g_n$ such that $[g|(\mathbb{R} \setminus \bigcup_{n<\omega} G_n \cap H_n)] + X$ is an SZ-set. This extension exists based on Corollary 9. Thus, $g + X$ is the union of countable many SZ-sets. Consequently, $g + X \in J_{SZ}$. Clearly, g intersects every blocking set, so g is almost continuous.

To complete the proof we need to show the above claim. Fix an $n < \omega$ and put $A_n = (G_n \cap H_n) \cup \bigcup_{I \in \mathcal{I}_n} I$. The set A_n is everywhere of second category. Notice also that the part of X contained in $(G_n \cap H_n) \times \mathbb{R}$ can be covered by n functions $f_1, \ldots, f_n$ from $\mathbb{R}$ to $\mathbb{R}$. So, by (∗), there exists a function $g_n: A_n \to \mathbb{R}$ such that $g_n + \{f_1, \ldots, f_n\} \subseteq$ SZ(A_n) and dom$(g_n \cap B)$ is dense in some non-empty open interval I_B for every blocking set B. Thus, if we define $g_n = g_n|(G_n \cap H_n)$ then $g_n + X \in J_{SZ}$.

What remains to prove is that $\bigcup_{n<\omega} g_n$ intersects every blocking set B. Notice that $I_B \cap G_n \neq \emptyset$ for some n. Thus, $g_n \cap B \neq \emptyset$. Consequently, $\emptyset \neq B \cap \bigcup_{n<\omega} g_n \subseteq B \cap g$. This finishes the proof. ∎

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